Robust Ordinal Regression

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Outline

1. Introduction
2. Robust ordinal regression for value-based model
3. Robust ordinal regression for outranking-based model
4. Robust ordinal regression for group decision
5. Extreme ranking analysis
6. Representative instance of the preference model
7. Different types of preference information
8. Summary
Focus on Particular Multiple Criteria Problem

Characteristics
- Digital economy (Economist Intelligence Unit in 2010)
- Quality of information and technology infrastructure
- European countries evaluated on 6 criteria
Multiple Criteria Decision Problems

**Characteristics**
- Actions evaluated on multiple criteria
- Family of criteria is supposed to satisfy the consistency conditions

**Ranking**
- Rank the actions from the best to the worst according to DM’s preferences
- Ranking can be complete or partial

**Choice**
- Choose a subset of the best actions
Multiple Criteria Decision Problems

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- Actions evaluated on multiple criteria
- Family of criteria is supposed to satisfy the consistency conditions

Sorting
- Pre-defined ordered classes
- Classes have a semantic definition
- Assignment to classes is grounded on absolute evaluation of actions
- No relative comparisons is required, ≠ choice, ranking
Multiple Criteria Decision Problems

Characteristics

- **Actions** evaluated on multiple criteria
- Family of criteria is supposed to satisfy the **consistency conditions**

Sorting

- **Pre-defined ordered** classes
- Classes have a **semantic definition**
- Assignment to classes is grounded on absolute evaluation of actions
- **No relative comparisons** is required, ≠ choice, ranking
Major Problems Related to MCDA

Elicitation of preference information & modeling of preferences
- The only objective information stemming from the problem formulation is the dominance relation in set $A$ - too poor to recommend a decision.
- To enrich the dominance relation, the DM has to elicit some preference information necessary to construct a model representing her value system.

Robustness analysis
- Examination of the impact of each model parameter on the final outcome.
- Influence of the input (preference ) information and imprecise evaluations of actions on variability of the proposed recommendation.
- Indication of the solutions which are good (bad) for different instances of a preference model.
Preference Model

Additive value function

- Allow compensation between criteria
- Criteria are supposed to be independent with respect to preferences
- Comprehensive (overall) value function:
  \[ U(a) = \sum_{j=1}^{m} u_j(g_j(a)) \]
  
  where the marginal value functions \( u_j \) are monotonic

- \( U \) is normalized so that \( U(a) \in [0, 1] \), for all \( a \in A \)
- Easy interpretation of numerical scores and straightforward translation into recommendation

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Outranking relation S groups three basic preference relations: \( S = \{\sim, \succ, \succsim\} \)

\( aSb \) means “action a is at least as good as action b”

Non-compensatory preference model used in the ELECTRE and PROMETHEE methods

Accept incomparability, no completeness nor transitivity

Outranking relation on set of actions A is constructed via concordance and discordance tests
Concordance and Discordance tests

**Concordance test**: Checks if the coalition of criteria concordant with the hypothesis \(aSb\) is strong enough

\[
C(a, b) = \frac{\sum_{j=1}^{m} k_j \cdot C_j(a, b)}{\sum_{j=1}^{m} k_j} = \frac{[k_1 C_1(a, b) + \ldots + k_m C_m(a, b)]}{(k_1 + \ldots + k_m)}
\]

Concordance test is **positive** if: \(C(a, b) \geq \lambda\), where \(\lambda\) is a cutting level (concordance threshold)

**Discordance test**: checks if among criteria discordant with the hypothesis \(aSb\) there is a strong opposition against \(aSb\):

\[
g_j(b) - g_j(a) \geq v_j \text{ (for gain-type criterion)}
\]

\[
g_j(a) - g_j(b) \geq v_j \text{ (for cost-type criterion)}
\]
Direct preference information

- Fixing a **precise numerical value** for the parameters (e.g., weights, discrimination thresholds, number of characteristic points)
- Aggregation paradigm: the model is first constructed and then applied on set $A$
- Requires great cognitive effort on the part of the DM
Elicitation of Preference Information

Indirect and imprecise preference information

- Examples of holistic judgments or imprecise information concerning parameters (supplied in a direct or indirect way)
- DMs are more easy with exercising decisions, rather than with explaining them in terms of preference model parameters
- Easy, natural, requires little cognitive effort, consistent with intuitive reasoning of the DM
Compatible instance of the preference model reproduces all exemplary decisions provided by the DM

- Usually, there exists more than one compatible instance
- Dealing with this indetermination is a major issue
Pairwise comparisons provided by the DM are reproduced.
Recommendation for non-reference actions may differ substantially when using different compatible instances.
Choice of a Single Compatible Instance Left to the DM

Interactive user controlled modification of graphically presented instances of the preference model

Required interpretation of the form of marginal value functions or marginal concordance/preference functions
Pre-defined Rules for Choice of a Single Instance

- Predefined rules aiming to obtain mean, average, central, or most discriminant instance
- Optimized criteria built only on the preference information
Robust Ordinal Regression (e.g., UTA$^{GMS}$ and GRIP)

Characteristics

- Take into account all instances of a preference model compatible with preference information
- Identify necessary and possible consequences of using all models:
  \[ a \succ^{N} b \iff \text{for all } U \in U^{AR}, U(a) \geq U(b) \]
  \[ a \succ^{P} b \iff \text{for at least one } U \in U^{AR}, U(a) \geq U(b) \]
Possible preference relation

Definition

\( a \succeq^P b \) means that \( U(a) \geq U(b) \) for at least one compatible value function

\[ a \succeq^P b \iff \varepsilon^* > 0 \text{ and } E(a, b) \text{ is feasible} \]

where:

\[ \varepsilon^* = \max \varepsilon \]

\[ U(a) \geq U(b) \]
\[ U(a^*) \geq U(b^*) + \varepsilon \iff a^* \succ b^* \text{ for } a^*, b^* \in A^R \]
\[ U(a^*) = U(b^*) \iff a^* \sim b^* \text{ for } a^*, b^* \in A^R \]
\[ u_j(x_j^k) - u_j(x_j^{(k-1)}) \geq 0, \ k = 2, \ldots, n_j(A), \ j = 1, \ldots, m \]
\[ u_j(x_j^1) = 0, \ j = 1, \ldots, m \]
\[ \sum_{j=1}^m u_j(x_j^{n_j(A)}) = 1 \]
Necessary preference relation

**Definition**

\( a \succeq^N b \) means that \( U(a) \geq U(b) \) for all compatible value functions

\[
\begin{align*}
    a \succeq^N b \iff & \quad \varepsilon_* \leq 0 \text{ or } E(b, a) \text{ is infeasible} \\
    & \\
    \text{where:} & \quad \varepsilon_* = \max \varepsilon \\
    U(b) & \geq U(a) + \varepsilon \\
    U(a^*) & \geq U(b^*) + \varepsilon \iff a^* \succ b^* \text{ for } a^*, b^* \in A^R \\
    U(a^*) & = U(b^*) \iff a^* \sim b^* \text{ for } a^*, b^* \in A^R \\
    u_j(x_j^k) - u_j(x_j^{(k-1)}) & \geq 0, \quad k = 2, \ldots, n_j(A), \quad j = 1, \ldots, m \\
    u_j(x_j^1) & = 0, \quad j = 1, \ldots, m \\
    \sum_{j=1}^m u_j(x_j^{n_j(A)}) & = 1
\end{align*}
\]

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Graph of the Necessary Relation
The scheme of Robust Ordinal Regression is general:
- preference information may have different forms
- preference model can also be different from the additive value function
Robust Ordinal Regression for Outranking Methods

**ELECTRE\textsuperscript{GKMS}**

- **Pairwise comparisons** of reference actions stating the truth or falsity of the outranking relation
- Intra-criterion preference information provided either indirectly or imprecisely
- ROR takes into account all instances of the outranking model compatible with the provided preference information
Robust Ordinal Regression for Outranking Methods

Characteristics

- **Compatible** outranking is able to restore all pairwise comparisons for provided imprecise intra-criterion preference information
- Infer inter-criteria parameters from pairwise comparisons
- Marginal concordance functions as **general** non-decreasing ones, defined in the “spirit” of ELECTRE methods
We assume that the weights of criteria sum up to one, i.e. $\sum_{j=1}^{m} k_j = 1$. Thus:

$$C(a, b) = \frac{\sum_{j=1}^{m} k_j \times C_j(a, b)}{\sum_{j=1}^{m} k_j} = \sum_{j=1}^{m} k_j \times C_j(a, b) = \sum_{j=1}^{m} \psi_j(a, b).$$

Set of concordance indices $C(a, b)$, cutting levels $\lambda$, indifference $q_j$, preference $p_j$, and veto thresholds $v_j$, $j = 1, \ldots, m$, satisfying the foll. set of constraints $E^A$:

If $aSb$ for $(a, b) \in B^R$

$$C(a, b) = \sum_{j=1}^{m} \psi_j(a, b) \geq \lambda$$
$$g_j(b) - g_j(a) + \varepsilon \leq v_j, \quad j = 1, \ldots, m$$

If $aS^C b$ for $(a, b) \in B^R$

$$C(a, b) = \sum_{j=1}^{m} \psi_j(a, b) + \varepsilon \leq \lambda + M_0(a, b)$$
$$g_j(b) - g_j(a) \geq v_j - \delta M_j(a, b)$$
$$M_j(a, b) \in \{0, 1\}, \quad j = 0, \ldots, m, \quad \sum_{j=0}^{m} M_j(a, b) \leq m$$

where $\delta$ is a big given value
ELECTRE\textsuperscript{GKMS} - Compatible outranking model (2)

1 \geq \lambda \geq 0.5, \quad v_j \geq p_j^* + \varepsilon

v_j \geq g_j(b) - g_j(a) + \varepsilon, \quad v_j \geq g_j(a) - g_j(b) + \varepsilon \quad \text{if} \quad a \sim_j b

Normalization:

\sum_{j=1}^{m} \psi_j(a_j^*, a_j, \ast) = 1

Monotonicity: for all \( a, b, c, d \in A \) and \( j = 1, \ldots, m \):

\begin{align*}
\psi_j(a, b) &\geq \psi_j(c, d) \quad \text{if} \quad g_j(a) - g_j(b) > g_j(c) - g_j(d) \\
\psi_j(a, b) &= \psi_j(c, d) \quad \text{if} \quad g_j(a) - g_j(b) = g_j(c) - g_j(d)
\end{align*}
ELECTRE\textsuperscript{GKMS} - Compatible outranking model (3)

partial concordance:  for all $(a, b) \in A \times A$ and $j = 1, \ldots, m$:

1. \( \psi_j(a, b) = 0 \) if \( g_j(a) - g_j(b) \leq -p_j^* \)
2. \( \psi_j(a, b) \geq \varepsilon \) if \( g_j(a) - g_j(b) > -p_j^* \)
3. \( \psi_j(a, b) + \varepsilon \leq \psi_j(a_j^*, a_j^*) \) if \( g_j(a) - g_j(b) < -q_j^* \)
4. \( \psi_j(a, b) = \psi_j(a_j^*, a_j^*) \) if \( g_j(a) - g_j(b) \geq -q_j^* \)

1. \( \psi_j(a, b) = 0 \) if \( b \succ_j a \)
4. \( \psi_j(a, b) = \psi_j(a_j^*, a_j^*), \, \psi_j(b, a) = \psi_j(a_j^*, a_j^*) \) if \( a \sim_j b \)
Robust Ordinal Regression for Outranking Methods

Recommendation

- Constructing four relations in the set of actions: necessary and possible outranking and non-outranking
- Space for interactivity
Matrix of the Necessary Outranking Relation (1)

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### Matrix of the Possible Outranking Relation (2)

The matrix below represents the possible outranking relations between different elements. Each cell indicates whether one element outranks another.

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## Matrix of the Necessary Outranking Relation (2)

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**Diagonal Elements:**  
- D: D, U, M, F, G, I, B, T, K, A  
- U: D, U, M, F, G, I, B, T, K, A  
- F: D, U, M, F, G, I, B, T, K, A  
- I: D, U, M, F, G, I, B, T, K, A  
- B: D, U, M, F, G, I, B, T, K, A  
- T: D, U, M, F, G, I, B, T, K, A  

**Non-Diagonal Elements:**  
- D: U, M, F, G, I, B, T, K, A  
- U: D, M, F, G, I, B, T, K, A  
- M: D, U, F, G, I, B, T, K, A  
- F: D, U, M, G, I, B, T, K, A  
- G: D, U, M, F, I, B, T, K, A  
- I: D, U, M, F, G, B, T, K, A  
- B: D, U, M, F, G, I, T, K, A  
- T: D, U, M, F, G, I, B, K, A  
- K: D, U, M, F, G, I, B, T, A  
Final Recommendation

Electre Is

Net Flow Score

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Robust Ordinal Regression for Sorting Methods

Preference information:
- \( \rightarrow [C_L(\circ), C_R(\circ)] \)
- \( \rightarrow [C_L(\circ), C_R(\circ)] \)
- + additional info

Preference model:
- All outranking models compatible with preference information

Apply on \( A \)

\[ L_P(\text{x}) \ldots R_P(\text{x}) \quad L_N(\text{x}) \ldots R_N(\text{x}) \]

- \( C_h \in [L_P(\text{x}), R_P(\text{x})] \), if for at least one \( S \in S_A^R \) we have \( \text{x} \rightarrow C_h \)
- \( C_h \in [L_N(\text{x}), R_N(\text{x})] \), if for all \( S \in S_A^R \) we have \( \text{x} \rightarrow C_h \)

ROR for multiple criteria sorting outranking methods
- Preference information: assignment examples
- Recommendation: necessary and possible assignments
- Assignment rule: \( aSb \Rightarrow C(a) \succ C(b) \)
- Implication: \( C(a) \succ C(b) \Rightarrow bS^C a \)
Possible and necessary assignments

C3
D U M

C2
B F

C1
G I

Possible assignment to C2
D U M
S^c
T
S^c
G I
All satisfied, then YES

Necessary assignment to C2
D U M
S
T
S
G I
At least one satisfied, then NO
Several DMs $\mathcal{D} = \{d_1, \ldots, d_s\}$ cooperate in a decision problem.

DMs share the same “description” of the decision problems.

The collective results (ranking or subset of the best actions) should account for preferences expressed by each DM.

Avoid discussions of DMs on technical parameters.

Reason in terms of necessary and possible relations and coalitions.
Results

For each $d_h \in D' \subseteq D$ who expresses her individual preferences, calculate the necessary and possible relation (or assignments)

With respect to all DMs four situations are considered:

- $a \succsim_{N,N} b : a \succsim_{d_h} b$ for all $d_h \in D'$
- $a \succsim_{N,P} b : a \succsim_{d_h} b$ for all $d_h \in D'$
- $a \succsim_{P,N} b : a \succsim_{d_h} b$ for at least one $d_h \in D'$
- $a \succsim_{P,P} b : a \succsim_{d_h} b$ for at least one $d_h \in D'$

Interdependencies between different types of results create space for interactivity

Dealing with incompatibility

- No instance of a preference model compatible with all pieces of preference information of all DMs
- Indicate which statements should be removed or revised
Consider preferences of DMs individually

- $A$ is small, DMs have outlook of the whole set $A$, interrelated preferences
- Analyze statements of DMs individually
- Examine the spaces of agreement and disagreement

Consider preferences of DMs simultaneously

- $A$ is numerous, DMs are experts only w.r.t. to its small disjoint subsets
- Combine knowledge of DMs into preference information of a single fictitious DM
Consider preferences of all DMs simultaneously

- Suppose that $\succeq^{AR}_D$ of compatible preference models is not empty
- One obtains two preference relations: $\succeq^N_D$ and $\succeq^P_D$
- Difference between $\succeq^N_D$ and $\succeq^{N,N}_D$
  - $a \succeq^N_D b \iff a \succeq b$ for all preference models compatible with all preferences of all DMs from $D$
  - $a \succeq^{N,N}_D b \iff a \succeq b$ for all compatible preference models of each DM from $D$
- If $\succeq^{AR}_D \neq \emptyset$, then for all $a, b \in A$
  - $a \succeq^{N,N}_D b \Rightarrow a \succeq^N_D b$ and $a \succeq^N_D b \Rightarrow a \succeq^{P,N}_D b$
Consider preferences of all DMs simultaneously

- Suppose that $\succeq^A_R$ of compatible preference models is **not empty**
- One obtains two preference relations: $\succeq^N_D$ and $\succeq^P_D$
- Difference between $\succeq^P_D$ and $\succeq^{P,P}_D$
  - $a \preceq^P_D b \iff a \succeq_P b$ for at least one preference model compatible with all preferences of all DMs from $D$
  - $a \preceq^{P,P}_D b \iff a \succeq_P b$ for at least one compatible preference model of at least one DM from $D$
- If $\succeq^A_R \neq \emptyset$, then for all $a, b \in A$,
  - $a \preceq^P_D b \Rightarrow a \succeq^{P,P}_D b$ and $a \preceq^P_D b \Rightarrow a \succeq^{P,N}_D b$
Necessary and possible binary relations vs. interpretation of DM

Complete rankings are intuitive, easy to understand, and popular

Very often, the DM is interested in ranks and scores of the actions
Extreme Ranking Analysis

- **the highest** $P^*(a)$ (rank) $P^*(a)$ **the lowest**

Assume that $a$ is ranked in the top in the bottom

Identify minimal subset of alternatives that are simultaneously not worse than $a$ not better than $a$

i.e., solve the following MILP problems:

$$\min: f^\text{pos}_\text{max} = \sum_{b \in A \setminus \{a\}} v_b$$

$$\min: f^\text{pos}_\text{min} = \sum_{b \in A \setminus \{a\}} v_b$$

$$\begin{cases} U(a) > U(b) - M \cdot v_b, \text{ for all } b \in A \setminus \{a\} \\ E(A^R) \end{cases}$$

$$\begin{cases} U(b) > U(a) - M \cdot v_b \\ E(A^R) \end{cases}$$

Read off the extreme ranks:

$$P^*(a) = f^\text{pos}_\text{max} + 1$$

$$P^*(a) = |A| - f^\text{pos}_\text{min}$$

Best case (Poland)

Worst case (Turkey)

Collate each action with all the remaining actions jointly

Compute the highest and the lowest ranks and scores
### Extreme Ranking Analysis

<table>
<thead>
<tr>
<th>Country</th>
<th>Rank</th>
<th>Rank</th>
</tr>
</thead>
<tbody>
<tr>
<td>Sweden</td>
<td>1</td>
<td>3</td>
</tr>
<tr>
<td>Netherland</td>
<td>1</td>
<td>4</td>
</tr>
<tr>
<td>Denmark</td>
<td>2</td>
<td>4</td>
</tr>
<tr>
<td>Norway</td>
<td>2</td>
<td>5</td>
</tr>
<tr>
<td>UK</td>
<td>3</td>
<td>8</td>
</tr>
<tr>
<td>France</td>
<td>4</td>
<td>9</td>
</tr>
<tr>
<td>Malta</td>
<td>4</td>
<td>9</td>
</tr>
<tr>
<td>Ireland</td>
<td>5</td>
<td>9</td>
</tr>
<tr>
<td>Germany</td>
<td>5</td>
<td>9</td>
</tr>
<tr>
<td>Slovakia</td>
<td>10</td>
<td>12</td>
</tr>
<tr>
<td>Poland</td>
<td>10</td>
<td>12</td>
</tr>
<tr>
<td>Turkey</td>
<td>10</td>
<td>13</td>
</tr>
<tr>
<td>Bulgaria</td>
<td>12</td>
<td>12</td>
</tr>
<tr>
<td>Russia</td>
<td>14</td>
<td>16</td>
</tr>
<tr>
<td>Kazakh.</td>
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<td>16</td>
</tr>
<tr>
<td>Ukraine</td>
<td>14</td>
<td>17</td>
</tr>
<tr>
<td>Azerbajian</td>
<td>16</td>
<td>17</td>
</tr>
</tbody>
</table>

- **Narrow ranges** (e.g., Bulgaria) vs. **wide ranges** (e.g., UK)
- **Interactive specification of pairwise comparisons** (e.g., (UK, Ireland), (Poland, Slovakia))
- **Choice of the best actions**, e.g., $\mathcal{BEST} = \{ a \in A : P^*(a) = 1 \}$

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“Flatten” consequences of applying all compatible instances
Reflect a reasonable compromise between all possible outcomes
Highlight the most stable parts of the results
Extend ROR in the capacity of explaining the final output
on the base of necessary and possible consequences of using preference information in the adopted preference model.
Targets to be Attained by the Representative Instance

- Targets concern enhancement of differences between scores of pairs of actions
- Emphasize the evident advantage of some actions over the others
- Reduce the ambiguity in the statement of the advantage
Targets consist in max or min of the difference between scores of actions satisfying the corresponding binary relation.

Targets can be combined into various scenarios according to the wish of the DM.
Specificity of Various Procedures

Type of problem | Number of DMs | Preference model
--- | --- | ---
$\alpha$ | $\beta$ | $\gamma$

Type of multiple criteria problem
- e.g., choice - emphasize the advantage of the best alternatives
- e.g., group decision - intensity connected with the inclusion relation on the set of all subsets of DMs

Underlying preference model
- e.g., check if concordance test is necessarily or possibly verified or failed, and control the difference between $C(a, b)$ and $\lambda$

Further exploitation of the representative instance
- e.g., recommendation at once (ranking in MAVT)
- e.g., example-based sorting or existing ELECTRE methods
Combine different types of preference information
- Flexibility: wide spectrum of preference information
- Zooming: possibility to represent preferences in a limited zone of the evaluation spaces of considered objectives
Desired Ranks of Actions

- should take place **on the podium**
- should (not) be ranked **among top / bottom 5 alternatives**
- should be among **the 10% of best / worst alternatives**
- is predisposed to secure **the place between 4 and 10**
- should be ranked **in the second ten of alternatives**
- should be ranked **in the upper / lower half of the ranking**

evaluation profile of 🎁 predisposes it **to have value at least / at most x**

where $x \in [0,1]$
- Adapted robust ordinal regression (necessary vs. possible)
- Adapted procedures for selection of a single value function
- Adapted extreme ranking analysis
a should be ranked among top r alternatives

1) \( U(a) + M \cdot v_{a,b}^> \geq U(b) + \varepsilon \), for all \( b \in A \setminus \{a\} \)

2) \( \sum_{b \in A \setminus \{a\}} v_{a,b}^> \leq r - 1 \)

Action a starts with rank “1” for “top” case (“n” for “bottom”)
If \( v_{a,b} = 1 \), corresponding constraint is always satisfied
Such relaxation is admitted for up to \( r - 1 \) constraints
Roman Słowiński  
Robust Ordinal Regression

Modeling of Rank Related Requirements

\( a \) should be ranked in a position in the range \([f, c]\), with \( f \leq c \)

\[ I) \quad U(a) + M \cdot v_{a,b}^{\geq} \geq U(b) + \varepsilon, \quad \text{for all } b \in A \setminus \{a\} \]

\[ II) \quad \sum_{b \in A \setminus \{a\}} v_{a,b}^{\geq} \leq c - 1 \]

\[ III) \quad U(b) + M \cdot v_{a,b}^{\leq} \geq U(a) + \varepsilon, \quad \text{for all } b \in A \setminus \{a\} \]

\[ IV) \quad \sum_{b \in A \setminus \{a\}} v_{a,b}^{\leq} \leq n - f \]

\[ V) \quad v_{a,b}^{\geq} + v_{a,b}^{\leq} \leq 1, \quad \text{for all } b \in A \setminus \{a\} \]

The most general statement

- At most \( c - 1 \) actions ranked not worse than \( a \)
- At most \( n - f \) actions ranked not better than \( a \)
- Subsets of actions which are ranked better or worse than \( a \) are disjoint

[f, c] = [6, 10]
Selection of a Single Value Function

<table>
<thead>
<tr>
<th>Ranking and comprehensive values</th>
<th>Marginal value functions</th>
</tr>
</thead>
<tbody>
<tr>
<td>Denmark 1</td>
<td>Slovakia 10</td>
</tr>
<tr>
<td>Sweden 2</td>
<td>Poland 10</td>
</tr>
<tr>
<td>UK 3</td>
<td>Bulgaria 12</td>
</tr>
<tr>
<td>Netherlan 4</td>
<td>Turkey 13</td>
</tr>
<tr>
<td>Norway 5</td>
<td>Russia 14</td>
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<tr>
<td>Malta 6</td>
<td>Kazakh. 15</td>
</tr>
<tr>
<td>France 7</td>
<td>Ukraine 15</td>
</tr>
<tr>
<td>Germany 8</td>
<td>Azerbaj. 17</td>
</tr>
<tr>
<td>Ireland 8</td>
<td></td>
</tr>
</tbody>
</table>

- Reference actions placed at positions which are in the range of desired ranks
- Desired “parts” of final ranking

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Actual range of attained ranks may be a proper subset of the range specified by the DM
Desired ranks vs. extreme ranking analysis
Summary

- Robust ordinal regression applied to value- and outranking-based methods
- Robust ordinal regression applied to group decision
- Extreme and representative results enrich the necessary and the possible ones
- Accounting for preference information of a new type in ranking and sorting problems
- Only a subset of methods has been covered during presentation
Selected Papers on Robust Ordinal Regression


Selected Papers on Robust Ordinal Regression


Robust Ordinal Regression

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April 15, 2013